# $p$-adic deformations of theta series and the Gross-Zagier formula

Henri Darmon (with Alan Lauder, Victor Rotger)

The Gross-Zagier formula, 40 years later
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# The broad theme of this lecture

*It is sometimes possible to "reverse engineer" the calculations of Gross-Zagier to obtain conjectural formulae for algebraic points on elliptic curves in settings where they cannot be shown to exist independently.*

Class number formula / Kronecker Limit Formula $\Rightarrow$ Stark units.

$$\Downarrow$$

Gross-Zagier Formula $\Rightarrow$ "Stark-Heegner points".

Stark-Heegner points are values of "rigid analytic cocycles" at special points, as described in Alice Pozzi's talk yesterday.

**Goal**: Relate them to fourier coefficients of the same type of modular generating series as appear in Gross-Zagier.

# The equivariant BSD conjecture

Let $E$ be an elliptic curve over $\mathbb{Q}$.

$W$ an Artin representation: $\varrho : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{Aut}(W) \simeq \mathrm{GL}_n(L)$

$E(H)^W := \hom_{G_\mathbb{Q}}(W, E(H) \otimes L), \qquad r(E, W) := \dim_L E(H)^W.$

$L(E, W, s)$ the Hasse-Weil-Artin $L$-series attached to $E$ and $W$.

**Equivariant Birch and Swinnerton-Dyer conjecture**:

$$r(E, W) = \mathrm{ord}_{s=1} L(E, W, s).$$

Progress on this conjecture has been painstaking and hard won.

# Results in "analytic rank zero"

If $L(E, W, 1) \neq 0$, then we know that $r(E, W) = 0$ when

• $W$ is one-dimensional (Kato $\sim$ 1991);

• $W$ is induced from a ring class character of an imaginary quadratic field (Gross-Zagier 1986, Kolvagin 1987, Bertolini–D 1998);

• $W$ is induced from a ring class character of a real quadratic field (Rotger–D, 2017);

• $W$ is the tensor product of two odd irreducible two-dimensional Galois representations (Rotger–D, 2017).

# Results in "analytic rank one"

They are confined to settings where $W$ is *self dual*,

$$W \simeq W^* = \mathrm{hom}_L(W, L),$$

so that

$$L(E, W, s) \leftrightarrow \mathrm{sign}(E, W) L(E, W, 2 - s),$$

with $\mathrm{sign}(E, W) = \pm 1$.

**Remark**: The self-duality requirement excludes all one dimensional representations except for quadratic characters.

**Conjecture**. Given an elliptic curve $E$ over $\mathbb{Q}$, there are finitely many Dirichlet characters $\chi$ of order $\geq 7$ with $L(E, \chi, 1) = 0$.

# Dihedral representations

Assume that the conductors of $E$ and $W$ are coprime.

$N := \mathrm{conductor}(E)$.

**Fact**: If $W$ is induced from a ring class character of a quadratic field $K$, then

$$\mathrm{sign}(E, W) = \varepsilon_K(-N).$$

• If $K$ is imaginary quadratic and $\varepsilon_K(-N) = -1$, the non-trivial element in $E(H)^W$ can be produced using a Heegner point construction, at least when $L'(E, W, 1) \neq 0$.

• If $K$ is real quadratic and $\varepsilon_K(N) = -1$, the provenance of the non-trivial element in $E(H)^W$ is more mysterious; this setting suggests the existence of *Stark-Heegner points* over ring class fields of real quadratic fields.

# Tensor products

Let $W_1$ and $W_2$ be two odd two dimensional Artin representations, attached to weight one cusp forms $g$ and $h$ with inverse nebentype characters.

**Equivariant BSD in analytic rank zero**: (Rotger, D, 2017). If $L(E, W_1 \otimes W_2, 1) \neq 0$, then $r(E, W_1 \otimes W_2) = 0$.

**Fact** When $\gcd(N, D) = 1$, then

$$\mathrm{sign}(E, W_1 \otimes W_2) = 1.$$

This setting therefore seems unpromising for revealing new Stark-Heegner point constructions.

It turns out to be fruitful to consider a twisted variant of the tensor product: the Asai, or tensor induction.

# Asai representations

$W_0$ = odd Artin representation of $G_F$ with $F$ a real quadratic field.

Choose $\tau_0 \in G_{\mathbb{Q}} - G_F$, and set $g' := \tau_0 g \tau_0^{-1}$.

**Induced representation**: $\mathrm{Ind}_F^{\mathbb{Q}} W_0 \simeq W_0 \oplus W_0$, with

$$g \cdot (v_1, v_2) = \begin{cases} (gv_1, g'v_2) & \text{if } g \in G_F; \\ (g\tau_0^{-1}v_2, \tau_0 g v_1) & \text{if } g \in G_{\mathbb{Q}} - G_F; \end{cases}$$

**Tensor induction**: $\mathrm{Ind}_F^{\mathbb{Q}} W_0 \simeq W_0 \otimes W_0$, with

$$g \cdot (v_1 \otimes v_2) = \begin{cases} gv_1 \otimes g'v_2 & \text{if } g \in G_F; \\ g\tau_0^{-1}v_2 \otimes \tau_0 g v_1 & \text{if } g \in G_{\mathbb{Q}} - G_F; \end{cases}$$

# The sign formula for Asai representations

**Fact**: Let $W_0$ be an odd two-dimensional Artin representation of a real quadratic field $F$, whose conductor is prime to $N(E)$, and let $W = \mathrm{Ind}^{\otimes}(W_0)$. If $\mathrm{Transfer}_F^{\mathbb{Q}}(\det W_0) = 1$, then $W$ is self-dual, and

$$\mathrm{sign}(E, W) = \varepsilon_F(N).$$

**Question**: When $\varepsilon_F(N) = -1$, construct the non-trivial elements of $E(H)^W$ whose existence is predicted by the equivariant BSD conjecture.

This is the motivating question behind the current project with Lauder and Rotger.

# Hilbert modular forms

Let $G$ be the Hilbert modular form attached to $W_0$.

$$\begin{array}{rcl} G & : & \mathrm{Ideals}(F) \longrightarrow L^\times, \\ G(\lambda) & := & \mathrm{Trace}(\mathrm{frob}_\lambda | W_0^{I_\lambda}), \\ G(\lambda^{n+1}) & := & G(\lambda)G(\lambda^n) - \chi(\lambda)G(\lambda^{n-1}), \\ G_{\mathfrak{M}}(\tau_1,\tau_2) & := & \displaystyle\sum_{\nu \in (\mathfrak{M}\delta_F^{-1})_+} G((\nu)\mathfrak{M}^{-1}\delta_F) q^\nu, \qquad q^\nu := e^{2\pi i(\nu_1\tau_1+\nu_2\tau_2)} \end{array}$$

$G_{\mathfrak{M}}(\tau_1,\tau_2)$ is of weight $(1,1)$ on

$$\Gamma_0(\mathfrak{D},\mathfrak{M}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a,d \in \mathcal{O}_F, \quad b \in \mathfrak{M}^{-1}, \quad c \in \mathfrak{M}\mathfrak{D} \right\}.$$

# Diagonal restrictions and Asai representations

The diagonal restriction of $G_{\mathfrak{M}}$ is intimately connected to the $L$-functions attached to the Asai representation $W$.

$$\Phi_{DM} := G_{\mathfrak{M}}(q,q), \qquad \Phi_M = \mathrm{Trace}_M^{DM}(\Phi_{DM}).$$

$\Phi_M$ is a classical modular form of weight 2 and level $M$.

**Theorem** For all $f \in M_2(\Gamma_0(M))$,

$$\langle f, \Phi_M \rangle_{\mathrm{Pet}} \sim L(f, W, 1) \mod^{\times} \bar{\mathbb{Q}}.$$

# A $p$-adic deformation of $\Phi_M$

$$G'_{\mathfrak{M}}(q_1, q_2) := \sum_{\substack{\nu \in (\mathfrak{M}\delta_F^{-1})_+ \\ p \nmid \nu}} \log(\nu) \cdot G((\nu)\mathfrak{M}^{-1}\delta_F) q^{\nu},$$

$$\Phi_{DM}^{(p)} = G'_{\mathfrak{M}}(q, q), \qquad \Phi_M^{(p)} = \mathrm{Trace}_M^{DM}(\Phi_{DM}^{(p)}).$$

$\Phi_M^{(p)}$ is a $p$-adic modular form of weight 2 and tame level $M$, which can be envisaged as a $p$-adic deformation of $\Phi_M$.

**Ordinary projection**: $e_{\mathrm{ord}} : M_2^{(p)}(\Gamma_0(M)) \longrightarrow M_2(\Gamma_0(Mp))$.

Write

$$e_{\mathrm{ord}}(\Phi_M^{(p)}) = \sum_f \mathscr{L}_{W,f} \cdot f \quad \text{(mod old forms)}.$$

## The Asai point attached to $W$

Let $f$ be a newform of level $Mp$, and let $A_f$ be the associated abelian variety quotient of $J_0(Mp)$, which has purely toric reduction at $p$. Over $\bar{\mathbb{Q}}_p$,

$$1 \longrightarrow \Lambda \longrightarrow T \longrightarrow A_f \longrightarrow 1, \qquad T \simeq \mathbb{G}_m^r.$$

A differential of the form $\omega := \eta^*(dt/t)$ with $\eta \in \mathrm{Hom}(T, \mathbb{G}_m)$ is called *toric*.

**Conjecture**. There is a global point

$$P_{\mathrm{Asai}}(W, f) \in \mathrm{Image}(A_f^W)$$

and a toric differential $\omega$ for which

$$\mathscr{L}_{W,f} = \log_\omega(P_{\mathrm{Asai}}(W, f)).$$

# The case where $G$ is a Hilbert Eisenstein series

Consider the case where

$$F = \mathbb{Q}(\sqrt{D}), \qquad G = E_{1,\chi},$$

is a Hilbert Eisenstein series of weight $(1,1)$ attached to a ring class character $\chi$.

$$W_0 = 1 \oplus \chi, \qquad W = 1 \oplus 1 \oplus \mathrm{Ind}_F^{\mathbb{Q}} \chi.$$

**Theorem** (D, Pozzi, Vonk 2023). The Asai point $P_{\mathrm{Asai}}(W, f)$ is equal to a Stark-Heegner point attached to $F$, conjecturally defined over the Hilbert class field of $F$.

The opening gambit of [DPV 2023] is suggested by the work of Gross-Zagier on factorisations of singular moduli, where the diagonal restriction of a holomorphic family of real analytic Eisenstein series plays a central role.

# The case where $G$ is a Hilbert theta series

**Judith Ludwig, Isabella Negrini, Alice Pozzi, Sandra Rozensztajn, and Hanneke Wiersema**.

$$F = \mathbb{Q}(\sqrt{D_1 D_2}), \qquad K = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}) = K_1 K_2,$$

and $G = \theta_\chi$ is a Hilbert theta series of weight $(1,1)$ attached to a finite order character $\chi$ of $K$.

$$W_0 = \mathrm{Ind}_K^F(\chi), \qquad W = \mathrm{Ind}_{K_1}^{\mathbb{Q}}(\chi_1) \oplus \mathrm{Ind}_{K_2}^{\mathbb{Q}}(\chi_2),$$

where $\chi_1 := \chi|_{\mathbb{A}_{K_1}^\times}, \qquad \chi_2 := \chi|_{\mathbb{A}_{K_2}^\times}.$

The Asai point $P_{\mathrm{Asai}}(W, f)$ is related to a Heegner/Stark-Heegner point attached to $K_1$ or $K_2$.

Cf. Alice's lecture of yesterday.

# The main question

The study undertaken in [LNPRW] suggests the following question:

**Question**. How generally can the Asai point $P_{\mathrm{Asai}}(W, f)$ be related to a Heegner or Stark-Heegner point arising from the values of rigid analytic cocycles at special points?

**Goal of the remainder of this lecture**: To sketch an outline of such an explicit relationship when $G = \theta_\chi$ is a theta series attached to a finite order character $\chi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ of a more general quadratic extension $K$ (of arbitrary signature) of a real quadratic field $F$.

## The Kudla-Millson theta kernel

Let $V$ be a quadratic space over $\mathbb{Q}$ of signature $(r, s)$.

Archimedean symmetric space:

$$\mathbb{D}_\infty = \{\text{negative } s\text{-planes in } V_\mathbb{R}\}, \qquad \dim \mathbb{D}_\infty = rs.$$

Each positive vector $v \in V_\mathbb{R}$ gives rise to a codimension $s$ cycle

$$Z_v := \{\Pi \subset V_\mathbb{R} \text{ with } \Pi \subset (\mathbb{R}v)^\perp\}.$$

**Definition**. The formal Kudla-Millson theta kernel attached to an integral lattice $L \subset V$ is the generating series

$$\tilde{\Theta}_L = \sum_{v \in L} Z_v \cdot q^{\langle v,v \rangle} \; = \; * + \sum_{n \geq 1} Z_n \cdot q^n \; \in \; \mathcal{Z}^s(\mathbb{D}_\infty)[\![q]\!].$$

# Modularity of the Kudla-Millson theta kernel

Let $\Gamma$ be the orthogonal group of $L$, and $\pi : \mathbb{D}_\infty \longrightarrow \Gamma\backslash\mathbb{D}_\infty$.

Because the cycles $Z_n$ are preserved by $\Gamma$, there exist

$$\bar{Z}_n \in \mathcal{Z}^s(\Gamma\backslash\mathbb{D}_\infty) \text{ satisfying } Z_n = \pi^*(\bar{Z}_n).$$

**Theorem** (Kudla-Millson) After projecting onto cohomology, the generating series

$$\Theta_L := \alpha_0 + \sum_{n\geq 1} [\bar{Z}_n] \cdot q^n \quad \in \quad H^s(\Gamma\backslash\mathbb{D}_\infty, \mathbb{Z})[\![q]\!]$$

is a modular form of weight $(r+s)/2$.

# The $p$-adic symmetric space of $V$

To $V$ is also associated a $p$-adic symmetric space

$$\mathbb{D}_p := \{\text{Isotropic lines in } V_{\mathbb{C}_p}\} - \bigcup_{\substack{v \in V_{\mathbb{Q}_p} \\ \langle v,v\rangle = 0}} (\mathbb{C}_p v)^{\perp}.$$

It is a rigid analytic space over $\mathbb{Q}_p$: $\mathbb{D}_p = \bigcup_m \mathbb{D}_p^{(m)}$, where

$$\mathbb{D}_p^{(m)} = \{\text{Isotropic lines in } V_{\mathbb{C}_p}\} - \bigcup_{\substack{v \in L_{\mathbb{Z}_p} \\ \langle v,v\rangle = 0}} \{\xi \in L_{\mathcal{O}_{\mathbb{C}_p}} \text{ s.t. } p^m | \langle \xi, v\rangle\}.$$

It is of dimension $r + s - 2$ and the $p$-adic analogue of $Z_v$,

$$Z_{v,p} = \{\xi \in \mathbb{D}_p \text{ s.t. } \langle v, \xi\rangle = 0\}$$

is of codimension 1.

# The $p$-adic Kudla-Millson theta kernel

For all $v \in V_{\mathbb{Q}_p}$, let $\omega_v := \frac{d\langle v,\xi\rangle}{\langle v,\xi\rangle} \in \Omega^1_{\mathrm{mer}}(\mathbb{D}_p)$.

**Definition**. The formal $p$-adic Kudla-Millson theta kernel attached to the lattice $L \subset V$ is the generating series

$$\tilde{\Theta}_{L,p} = \sum_{v \in L_{\mathrm{prim}}} Z_v \cdot \omega_v \cdot q^{\langle v,v\rangle} = \sum_n R_n \cdot q^n \;\; \in \;\; \mathcal{Z}^s(\mathbb{D}_\infty) \otimes \Omega^1_{\mathrm{mer}}(\mathbb{D}_p)[\![q]\!].$$

Since $R_n \in \mathcal{Z}^s(\mathbb{D}_\infty) \otimes \Omega^1_{\mathrm{mer}}(\mathbb{D}_p)$ is preserved by $\Gamma$, it gives rise to a class $[R_n] \in H^s(\Gamma, \Omega^1_{\mathrm{mer}}(\mathbb{D}_p))$.

**Definition**. The $p$-adic Kudla-Millson series attached to the lattice $L \subset V$ is the generating series

$$\Theta_{L,p} = \sum_n [R_n] \cdot q^n \quad \in \quad H^s(\Gamma, \Omega^1_{\mathrm{mer}}(\mathbb{D}_p))[\![q]\!].$$

# Modularity of the $p$-adic Kudla-Millson series

**Conjecture**. The $p$-adic Kudla-Millson series is an overconvergent $p$-adic modular form of weight $(r+s)/2$ with coefficients in $H^s(\Gamma, \Omega^1_{\mathrm{mer}}(\mathbb{D}_p))$.

**Idea behind this conjecture**. $p$-adic families of Funke-Millson theta series (Lennart Gehrmann, Paul Kiefer):

$$\tilde{\Theta}^{(r)}_{L,p} = \sum_{v \in L_{\mathrm{prim}}} Z_v \cdot \langle v, \xi \rangle^r \cdot q^{\langle v, v \rangle} \quad \in \quad \mathcal{Z}^s(\mathbb{D}_\infty) \otimes H^0(\mathbb{D}_p^{(0)}, \mathscr{L}^{\otimes r})[\![q]\!].$$

$$d_\xi \left( \left( \frac{d}{dr} \tilde{\Theta}^{(r)}_{L,p} \right)_{r=0} \right) = \tilde{\Theta}_{L,p},$$

where $d_\xi : \frac{H^0(\mathbb{D}_p^{(0)}, \mathcal{O}_{\log})}{\mathbb{C}_p} \longrightarrow H^0(\mathbb{D}_p^{(0)}, \Omega^1(\mathbb{D}_p^{(0)}))$.

# The ordinary projection

**Elementary observation**: The class $[R_{np^m}] \in H^s(\Gamma, \Omega^1_{\mathrm{mer}}(\mathbb{D}_p))$ is analytic on the affinoid $\mathbb{D}_p^{(m)}$.

Hence $U_p^m(\Theta_{L,p})$ belongs to $H^s(\Gamma, \Omega^1(\mathbb{D}_p^{(m)}))[\![q]\!]$.

Let $\Gamma[p]$ be the $p$-arithmetic group preserving $L[1/p]$.

**Conjecture** The classes $[R_{np^m}]$ converge to a class $J_n \in H^s(\Gamma, \Omega^1(\mathbb{D}_p))$ as $m \to \infty$. This class extends to $\Gamma[p]$.

**Corollary**: The ordinary projection

$$\Theta^\circ_{L,p} := e_{\mathrm{ord}}\Theta_{L,p} = \sum_n J_n \cdot q^n \quad \in \quad H^s(\Gamma[p], \Omega^1(\mathbb{D}_p))[\![q]\!]$$

is a classical modular form of weight $(r+s)/2$ with coefficients in $H^s(\Gamma[p], \Omega^1(\mathbb{D}_p))$. It is called the $p$-adic Kudla-Millson theta kernel associated to $L$.

## A theta lifting to rigid analytic cocycles

The $p$-adic Kudla-Millson theta kernel $\Theta^{\circ}_{L,p}$ realises a theta correspondence between classical modular forms of weight $(r+s)/2$ and rigid analytic classes in $H^s(\Gamma[p], \Omega^1(\mathbb{D}_p))$.

Given a classical eigenform $f$ of weight $(r+s)/2$, let

$$J_f \in H^s(\Gamma, \Omega^1(\mathbb{D}_p)) \otimes K_f$$

be the rigid analytic cocycle associated to it by this theta correspondence.

In the most interesting case where $s \geq 1$, the class $J_f$ lifts essentially uniquely to a class $\Phi_f \in H^s(\Gamma, \mathcal{O}_{\log}(\mathbb{D}_p))$ satisfying

$$d\Phi_f = J_f.$$

# Relation with $P_{\mathrm{Asai}}(W, f)$

Suppose that $W_\psi$ is the Asai induction of a character $\psi$ of a quartic extension $K/F/\mathbb{Q}$.

Recall the conjectural global point $P_{\mathrm{Asai}}(W_\psi, f)$ attached to $W$.

Each fractional ideal $\mathfrak{a}$ of $K$ gives rise to a lattice $L_{\mathfrak{a}}$ in the quadratic space $V = K$ equipped with the form

$$q(x) = \mathrm{Tr}_{\mathbb{Q}}^{F}\mathrm{Norm}_{F}^{K}(x),$$

and endowed with the natural left multiplication action of $K_1^\times$.

# Special cycles and special points attached to $K$

To the data of $\mathfrak{a} \subset K$ we can associate the following structures:

• At $\infty$: an $s$-dimensional cycle $\Delta_{\mathfrak{a}} \in \mathcal{Z}_s(\Gamma \backslash \mathbb{D}_\infty)$.

• At $v = p$, the $K_1^\times$-action on $V_{\mathbb{C}_p}$ gives rise to 4 eigenlines $\xi_{\mathfrak{a},1}, \xi'_{\mathfrak{a},1}, \xi_{\mathfrak{a},2}, \xi'_{\mathfrak{a},2}$ which belong to $\mathbb{D}_p$ when $p$ is inert in $K$, and can be envisaged as "special points" on this $p$-adic symmetric domain.

The pair $(K, \psi)$ gives rise to a "reflex pair" $(K^*, \psi^*)$ satisfying

$$\mathrm{Ind}_F^{\otimes \mathbb{Q}}(\mathrm{Ind}_K^F \psi) = \mathrm{Ind}_{K^*}^{\mathbb{Q}}(\psi^*),$$

along with a map $\mathfrak{a} \mapsto \mathfrak{a}^*$ from the fractional ideals of $K$ to those of $K^*$.

# The main result

**Main formula**: (Lauder, Rotger, D, in progress).

$$\log_{\omega}(P_{\mathrm{Asai}}(W_{\psi}, f)) = \sum_{\mathfrak{a}} \psi^*(\mathfrak{a}^*) \left( (\Phi_f \cdot \Delta_{\mathfrak{a}})(\xi_{\mathfrak{a},1}) + (\Phi_f \cdot \Delta_{\mathfrak{a}})(\xi'_{\mathfrak{a},1}) \right).$$

This expresses the Asai point $P_{\mathrm{Asai}}(W_{\psi}, f)$ as a combination of special values of rigid analytic cocycles.

# A refined conjecture

**Conjecture**: The individual terms in the right hand sum are logarithms of global points defined over class fields of $K^*$: $(\Phi_f \cdot \Delta_{\mathfrak{a}})(\xi_{\mathfrak{a},1})$ is the $p$-adic logarithm of a global point on $E$ defined over an abelian extension of the reflex field $K^*$ (which is also the field cut out by the Asai representation $W$).

**Key remark**: When $(r, s) = (2, 2)$ (resp. $(3, 1)$), the spin group of $V$ is isomorphic to an inner form of $\mathrm{SL}_2(\mathcal{O}_{F^*})$ where $F^*$ is a real (resp imaginary) quadratic field.

The special values of rigid analytic cocycles for the Bianchi group have been studied numerically by Mak Trifkovic in 2006 and seem to parametrise elliptic curves over imaginary quadratic fields. Our $O(3, 1)$-classes are special cases of those considered by Trifkovic, arising from images under a Doi-Naganuma style "base change lift".

# Benedict Gross (1950-2025)

This lecture is dedicated to the memory of Dick Gross.

A great many thanks to the organisers of this wonderful conference for giving all of us such a chance to reflect on the enduring influence of Dick's mathematical ideas.